THREE PATHS TO THE RANK METRIC

Base camp

M. Ceria
Politecnico di Bari

Recap on Hamming metric codes

Linear

$C \leq (\mathbb{F}_q)^n$, $\dim_{\mathbb{F}_q}(C) = k$.

Hamming Distance

$\overline{\mathbf{x}} = (x_1, ..., x_n), \overline{\mathbf{y}} = (y_1, ..., y_n) \in C$:

$$d^H(\overline{\mathbf{x}}, \overline{\mathbf{y}}) = |\{i : \ x_i \neq y_i\}|$$

Weight

$$w^H(\overline{\mathbf{x}}) = |\{i : \ x_i \neq 0\}|$$

Recap on Hamming metric codes

Minimum distance

$$d^H(C) := \min\{d^H(\overline{\mathbf{x}}, \overline{\mathbf{y}}) : \overline{\mathbf{x}}, \overline{\mathbf{y}} \in C, \overline{\mathbf{x}} \neq \overline{\mathbf{y}}\} = \min\{w^H(\overline{\mathbf{x}}) : \overline{\mathbf{x}} \in C, \overline{\mathbf{x}} \neq \overline{\mathbf{0}}\}$$

$[n, k, d]_q$ code

Singleton bound

$$d \leq n - k + 1$$

MDS if equality.

Recap on Hamming metric codes

Support

$$\mathrm{Supp}^H(\overline{\mathbf{x}}) = \{i : x_i \neq 0\}$$

$$\mathrm{Supp}^H(C) = \bigcup_{\overline{\mathbf{v}} \in C} \mathrm{Supp}(\overline{\mathbf{v}})$$

Non-degenerate

$\mathrm{Supp}^H(C) = \{1, ..., n\}$; (degenerate otherwise)

r-th generalized weight

C non-degenerate, $r = 1, ..., k$:

$$w_r^H(C) = \min\{|\mathrm{Supp}^H(V)|, \ V \leq C, \dim(V) = r\}.$$

MINIMAL WORD $\overline{\mathbf{c}} \in C$

if $\overline{\mathbf{c}}' \neq \overline{\mathbf{0}}$ with $\mathrm{Supp}^H(\overline{\mathbf{c}}') \subset \mathrm{Supp}^H(\overline{\mathbf{c}})$, then $\overline{\mathbf{c}}'$ is a multiple of $\overline{\mathbf{c}}$.

MINIMAL CODE

all words are minimal.

Recap on Hamming metric codes

Generator matrix

$G \in M_{k,n}(\mathbb{F}_q)$: the rows are a basis of C as a vector space over $\mathbb{F}_q$.

Monomially equivalent Hamming metric codes

$C \sim C'$ if there is $f : (\mathbb{F}_q)^n \to (\mathbb{F}_q)^n$, an $\mathbb{F}_q$-linear isometry such that $C' = f(C)$

Dual code

$C^{\perp} = \{\overline{\mathbf{x}} \in (\mathbb{F}_q)^n : \overline{\mathbf{x}} \cdot \overline{\mathbf{y}} = 0, \forall \overline{\mathbf{y}} \in C\}$

$[n, n-k]$ code

If G is the generator matrix of C then $G\overline{\mathbf{x}}_T = 0, \forall \overline{\mathbf{x}} \in C^{\perp}$ (G is the parity-check matrix).

Textbook

Elisa Gorla

Rank-metric codes.

In Concise Encyclopedia of Coding Theory (pp. 227-250).

Chapman and Hall/CRC (2021).

Rank metric codes – matrices

(M)RMC – Delsarte

$\mathcal{C} \subseteq M_{n,m}(\mathbb{F}_q)$, different from the empty set.

(M)RMC – linear

$\mathcal{C} \leq M_{n,m}(\mathbb{F}_q)$, $\mathbb{F}_q$-linear subspace.

Rank distance

$$
\begin{aligned}
d: & M_{n,m}(\mathbb{F}_q) \times M_{n,m}(\mathbb{F}_q) \to \mathbb{N} \\
& (M, N) \mapsto r(M - N)
\end{aligned}
$$

where r is the **rank** of the matrix, which is for us the **weight function**.

VRMC – GABIDULIN

$C \leq (\mathbb{F}_{q^m})^n$, vector subspace over $\mathbb{F}_{q^m}$.

RANK DISTANCE

$$\begin{aligned} d :& (\mathbb{F}_{q^m})^n \times (\mathbb{F}_{q^m})^n \to \mathbb{N} \\ & (\overline{\mathbf{v}}, \overline{\mathbf{w}}) \mapsto r(\overline{\mathbf{v}} - \overline{\mathbf{w}}) \end{aligned}$$

RANK WEIGHT

For $\overline{\mathbf{v}} \in (\mathbb{F}_{q^m})^n$, $\overline{\mathbf{v}} = (v_1, \ldots, v_n)$, $v_i \in \mathbb{F}_{q^m}$, $1 \leq i \leq n$.

$$r(\overline{\mathbf{v}}) = \dim_{\mathbb{F}_q}(\langle v_1, \ldots, v_n \rangle)$$

METRICS

$(M_{n,m}(\mathbb{F}_q), r)$ is a metric space.

$((\mathbb{F}_{q^m})^n, r)$ is a metric space.

And we will see soon that there is a strong link between the two of them.

From VRMC to MRMC

Let $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ be a basis of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$.
Take $\bar{\mathbf{v}} = (v_1, \ldots, v_n) \in (\mathbb{F}_{q^m})^n$.

We define a matrix $\Gamma(\bar{\mathbf{v}}) \in M_{n,m}(\mathbb{F}_q)$, so that, for $1 \leq i \leq n$,

$$v_i = \sum_{j=1}^{m} \Gamma(\bar{\mathbf{v}})_{ij}\gamma_j$$

VRMC $\rightarrow$ MRMC

$C \leq (\mathbb{F}_{q^m})^n$ a VRMC, the MRMC associated to C wrt. Γ is

$$\Gamma(C) = \{\Gamma(\bar{\mathbf{v}}) : \bar{\mathbf{v}} \in C\}$$

Example

$\mathbb{F}_2 \subseteq \mathbb{F}_8 = \mathbb{F}_2(\alpha)$, $\alpha^3 = \alpha + 1$.

$\Gamma = \{1, \alpha, \alpha^2\}$

$C = \langle(\alpha, 1)\rangle$; $dim_{\mathbb{F}_8}(C) = 1$

$r((\alpha, 1)) = 2$

Example

$\mathbb{F}_2 \subseteq \mathbb{F}_8 = \mathbb{F}_2(\alpha)$, $\alpha^3 = \alpha + 1$.
$\Gamma = \{1, \alpha, \alpha^2\}$
$C = \langle(\alpha, 1)\rangle$

$$\Gamma(C) = \langle \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \rangle, \ dim(\Gamma(C)) = 3.$$

On this "mapping"

$$\begin{aligned} \Gamma : C &\to \Gamma(C) \\ \bar{\mathbf{v}} &\mapsto \Gamma(\bar{\mathbf{v}}) \end{aligned}$$

Properties

1. $\mathbb{F}_q$-linear map;
2. bijection;
3. preserves the rank: $r(\Gamma(\bar{\mathbf{v}})) = r(\bar{\mathbf{v}})$;
4. if $C \leq \mathbb{F}_{q^m}$ VRMC, $\dim_{\mathbb{F}_{q^m}}(C) = k$, then $\Gamma(C) \leq M_{n,m}(\mathbb{F}_q)$ MRMC, $\dim_{\mathbb{F}_q}(\Gamma(C)) = km$.

Going back and forth

$C \leq (\mathbb{F}_{q^m})^n$ VRMC, $\mathbb{F}_{q^m}$-linear $\rightarrow \Gamma(C) \leq M_{n,m}(\mathbb{F}_q)$ MRMC, $\mathbb{F}_q$-linear.

$\mathcal{C} \leq M_{n,m}(\mathbb{F}_q)$ MRMC: if you map it back into $(\mathbb{F}_{q^m})^n$ you **may loose linearity.**

Example

$\mathbb{F}_2 \subseteq \mathbb{F}_8 = \mathbb{F}_2(\alpha)$, $\alpha^3 = \alpha + 1$.
$\Gamma = \{1, \alpha, \alpha^2\}$

$$C = \langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \rangle = \left\{ O_{2,3}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right\}$$

Mapping back we get $\{\bar{\mathbf{0}}, (1, \alpha)\} \subseteq (\mathbb{F}_8)^2$ that is not linear over $\mathbb{F}_8$.

$\mathbb{F}_q$-linear isometry of $M_{n,m}(\mathbb{F}_q)$ wrt the rank metric

$$\phi : M_{n,m}(\mathbb{F}_q) \to M_{n,m}(\mathbb{F}_q)$$

- homomorphism over $\mathbb{F}_q$;
- $r(M) = r(\phi(M)), \forall M \in M_{n,m}(\mathbb{F}_q)$.

$\mathbb{F}_{q^m}$-LINEAR ISOMETRY OF $(\mathbb{F}_{q^m})^n$ WRT THE RANK METRIC

$$\psi : (\mathbb{F}_{q^m})^n \to (\mathbb{F}_{q^m})^n$$

- $\mathbb{F}_{q^m}$-linear homomorphism;
- $r(\overline{\mathbf{v}}) = r(\psi(\overline{\mathbf{v}}))$, $\forall \overline{\mathbf{v}} \in (\mathbb{F}_{q^m})^n$.

Characterization $\mathbb{F}_q$-linear isometries of $M_{n,m}(\mathbb{F}_q)$

Hua – Wan

$$\phi : M_{n,m}(\mathbb{F}_q) \to M_{n,m}(\mathbb{F}_q)$$

$\mathbb{F}_q$-linear isometry of $M_{n,m}(\mathbb{F}_q)$ wrt. the rank metric

$m \neq n$
$\exists A \in GL_n(\mathbb{F}_q)$, $B \in GL_m(\mathbb{F}_q)$:

$$\forall M \in M_{n,m}(\mathbb{F}_q), \quad \phi(M) = AMB.$$

Characterization $\mathbb{F}_q$-linear isometries of $M_{n,m}(\mathbb{F}_q)$

$$\phi : M_{n,m}(\mathbb{F}_q) \to M_{n,m}(\mathbb{F}_q)$$

$\mathbb{F}_q$-linear isometry of $M_{n,m}(\mathbb{F}_q)$ wrt. the rank metric

$m = n$

$\exists A, B \in GL_n(\mathbb{F}_q)$:

$$\forall M \in M_{n,m}(\mathbb{F}_q), \quad \phi(M) = AMB$$

or

$$\forall M \in M_n(\mathbb{F}_q), \quad \phi(M) = AM_T B$$

Characterization $\mathbb{F}_{q^m}$-linear isometries of $(\mathbb{F}_{q^m})^n$

Berger

$$\psi : (\mathbb{F}_{q^m})^n \to (\mathbb{F}_{q^m})^n$$

$\mathbb{F}_{q^m}$-linear isometry of $(\mathbb{F}_{q^m})^n$ wrt. the rank metric

$\exists \alpha \in \mathbb{F}^*_{q^m}, B \in GL_n(\mathbb{F}_q)$:

$$\forall \bar{\mathbf{v}} \in (\mathbb{F}_{q^m})^n, \quad \psi(\bar{\mathbf{v}}) = \alpha \bar{\mathbf{v}} B$$

Equivalence of MRMC

$\mathcal{C}, \mathcal{D} \leq M_{n,m}(\mathbb{F}_q)$ MRMC are *linearly equivalent*,

$$\mathcal{C} \sim \mathcal{D},$$

if there is

$$\phi : M_{n,m}(\mathbb{F}_q) \to M_{n,m}(\mathbb{F}_q)$$

$\mathbb{F}_q$-linear isometry of $M_{n,m}(\mathbb{F}_q)$:

$$\phi(\mathcal{C}) = \mathcal{D}.$$

$\mathbb{F}_{q^m}$-linear MRMC (GJLR)

$\mathcal{C} \leq M_{n,m}(\mathbb{F}_q)$ s.t. you can find a VRMC $C \leq (\mathbb{F}_{q^m})^n$ and a basis Γ of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$ such that $\mathcal{C} \sim \Gamma(C)$.

Equivalence of VRMC

$C, D \leq (\mathbb{F}_{q^m})^n$ VRMC are *linearly equivalent*,

$$C \sim D,$$

if there is

$$\psi : (\mathbb{F}_{q^m})^n \to (\mathbb{F}_{q^m})^n$$

$\mathbb{F}_{q^m}$-linear isometry of $(\mathbb{F}_{q^m})^n$:

$$\phi(C) = D.$$

The same equivalence?

We know how to pass from VRMCs to MRMCs; is the notion of **equivalence compatible** with this passage?

Yes!! (GJLR)

$C, D \leq (\mathbb{F}_{q^m})^n$ VRMC and Γ, Γ' bases of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$.

$$C \sim D \Rightarrow \Gamma(C) \sim \Gamma'(D).$$

The isometry group

$\mathcal{G}(q, m, n) \simeq \mathbb{F}_{q^m}^* \times GL_n(\mathbb{F}_q)$

Right action on $(\mathbb{F}_{q^m})^n$

$$\begin{aligned} \mathcal{G}(q, m, n) \times (\mathbb{F}_{q^m})^n &\to (\mathbb{F}_{q^m})^n \\ ((\alpha, M), \overline{\mathbf{v}}) &\mapsto \alpha \overline{\mathbf{v}} M \end{aligned}$$

$C, C' \leq (\mathbb{F}_{q^m})^n, \quad C \sim C' \Leftrightarrow \exists A \in GL_n(\mathbb{F}_q) : C' = CA := \{\overline{\mathbf{v}}A : \overline{\mathbf{v}} \in C\}$

MacWilliams Extension Theorem

Let $\mathcal{C}, \mathcal{D} \leq M_{n,m}(\mathbb{F}_q)$ two MRMC and suppose to have a $\mathbb{F}_q$-linear isometry between the two, can we extend it to the ground space?

The answer is **negative**.

Counterexamples

A. Barra and H. Gluesing-Luerssen, MacWilliams Extension Theorems and the Local-Global Property for Codes over Frobenius Rings, Journal of Pure and Applied Algebra 219 (2015), 703–728

J. de la Cruz, M. Kiermaier, A. Wassermann, and W. Willems, Algebraic structures of MRD codes, Advances in Mathematics of Communications, 10(3), 2016.

SUPPORT – VRMC

R. Jurrius and R. Pellikaan, On defining generalized rank weights, Advances in Mathematics of Communications 11 (2017), 225–235.

$C \leq (\mathbb{F}_{q^m})^n$ VRMC; $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ a basis of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$:

$$supp(\overline{\mathbf{v}}) = colsp(\Gamma(\overline{\mathbf{v}})) \leq (\mathbb{F}_q)^n$$

is the (rank) **support** of $\overline{\mathbf{v}} \in C$, and it is a $\mathbb{F}_q$-linear space.

ABNR

Rank weight: $r(\overline{\mathbf{v}}) = dim_{\mathbb{F}_q}(supp(\overline{\mathbf{v}}))$

$D \leq C \leq (\mathbb{F}_{q^m})^n$

$$supp(D) = \sum_{v \in D} supp(\overline{\mathbf{v}}) \leq (\mathbb{F}_q)^n.$$

Properties of the support (GJLR)

Take a vector $\bar{\mathbf{v}} \in (\mathbb{F}_{q^m})^n$.

For each $\alpha \in \mathbb{F}^*_{q^m}$, for each basis Γ of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$

$$supp(\alpha\bar{\mathbf{v}}) = supp(\bar{\mathbf{v}}).$$

The support does **not depend** on the choice made on the **basis**.

For every $A \in M_n(\mathbb{F}_q)$,

$$\Gamma(\bar{\mathbf{v}}A) = A_T\Gamma(\bar{\mathbf{v}})$$

PROPERTIES OF THE SUPPORT (ABNR)

We know that

$$supp(C) = \sum_{\overline{\mathbf{v}} \in C} supp(\overline{\mathbf{v}})$$

Being

$$supp(\overline{\mathbf{v}} + \overline{\mathbf{w}}) \leq supp(\overline{\mathbf{v}}) + supp(\overline{\mathbf{w}})$$

$C = \langle \overline{\mathbf{c}}_1, ..., \overline{\mathbf{c}}_t \rangle \leq (\mathbb{F}_{q^m})^n$ VRMC:

$$supp(C) = supp(\overline{\mathbf{c}}_1) + ... + supp(\overline{\mathbf{c}}_t).$$

Non-degenerate code

$C \leq (\mathbb{F}_{q^m})^n$ VRMC with length n and dimension k.

It is **nondegenerate** if

$$\mathrm{Supp}(C) = \mathbb{F}_q^n$$

Effettive length

$$\dim(\mathrm{Supp}(C))$$

Support – MRMC

$\mathcal{C} \leq M_{n,m}(\mathbb{F}_q)$ MRMC and $M \in \mathcal{C}$:

$n \leq m$

$$supp(M) = colsp(M) \leq (\mathbb{F}_q)^n$$

$n > m$

$$supp(M) = rowsp(M) \leq (\mathbb{F}_q)^m$$

SUBCODE SUPPORTED ON A SUBSPACE

$\mathcal{C} \leq M_{n,m}(\mathbb{F}_q)$, $h := min\{n, m\}$ $J \leq (\mathbb{F}_q)^h$.

Subcode supported on $J \leq (\mathbb{F}_q)^h$:

$$\mathcal{C}(J) := \{M \in \mathcal{C} : \; supp(M) \leq J\}.$$

MRMC: Minimum distance – Maximum Rank

Minimum distance

$\mathcal{C} \leq M_{n,m}(\mathbb{F}_q)$ MRMC

- $\mathcal{C} \neq 0$:

$$d_{min}(\mathcal{C}) = \min\{r(M) : M \in \mathcal{C}, M \neq O_{nm}\}$$

- $\mathcal{C} = 0$:

$$d_{min}(\mathcal{C}) = d_{min}(0) = \min\{n, m\} + 1.$$

Maximum Rank

$$\mathrm{maxr}(\mathcal{C}) := \max\{r(M) : M \in \mathcal{C}\}.$$

VRMC: Minimum distance – Maximum Rank

Minimum distance

$C \leq (\mathbb{F}_{q^m})^n$ VRMC

- $C \neq 0$:

$$d_{min}(C) = \min\{r(\overline{\mathbf{v}}) : \overline{\mathbf{v}} \in C, \overline{\mathbf{v}} \neq \overline{\mathbf{0}}\}$$

- $C = 0$:

$$d_{min}(C) = d_{min}(0) = n + 1.$$

Maximum Rank

$$\mathrm{maxr}(C) := \max\{r(\overline{\mathbf{v}}) : \overline{\mathbf{v}} \in \mathcal{C}\}.$$

$C \neq 0$

$$d_{min}(C) = d_{min}(\Gamma(C))$$

NOTE THAT (ABNR)

$$d_{min}(C) \leq d^H(C).$$

For $\bar{\mathbf{v}} \in (\mathbb{F}_{q^m})^n$, $\bar{\mathbf{v}} = (v_1, \ldots, v_n)$, $\bar{\mathbf{i}} \in \mathbb{F}_{q^m}$, $1 \leq i \leq n$.

$$r(\bar{\mathbf{v}}) = \min\{w^H(\bar{\mathbf{v}}A) : A \in GL_n(q)\}$$

Bounds for MRMC

$\mathcal{C} \leq M_{n,m}(\mathbb{F}_q)$ MRMC

Singleton bound (Delsarte)

$$\dim_{\mathbb{F}_q}(\mathcal{C}) \leq \max\{n, m\}(\min\{n, m\} - d_{\min}(\mathcal{C}) + 1)$$

Equality: **MRD** (max. rank. distance).

Anticode bound (Meshulam-Ravagnani)

$$\dim_{\mathbb{F}_q}(\mathcal{C}) \leq \max\{n, m\}\mathrm{maxr}(\mathcal{C})$$

Equality: **Optimal Anticode**.

Bounds for VRMC

$C \leq (\mathbb{F}_{q^m})^n$ VRMC

Singleton bound (Gabidulin)

$$\dim_{\mathbb{F}_{q^m}}(C) \leq n - d_{\min}(C) + 1$$

Equality: **MRD** (max. rank. distance).

Anticode bound (Ravagnani)

$\dim_{\mathbb{F}_{q^m}}(C) \leq m$:

$$\dim_{\mathbb{F}_{q^m}}(C) \leq \mathrm{maxr}(C)$$

Equality: **Optimal Vector Anticode**.

If $\dim_{\mathbb{F}_{q^m}}(C) > m$ it is not an optimal vector anticode

$$\dim_{\mathbb{F}_{q^m}}(C) > m \geq \mathrm{maxr}(C).$$

If $n > m$ the only MRD VRMC are $0, (\mathbb{F}_{q^m})^n$

For each $C \leq (\mathbb{F}_{q^m})^n$ optimal vector anticode

$$\dim_{\mathbb{F}_{q^m}}(C) = \mathrm{maxr}(C) \leq \min\{m, n\}$$

so if $m < n$ $(\mathbb{F}_{q^m})^n$ is not optimal vector anticode.

Adjoint code

$\mathcal{C} \leq M_{n,m}(\mathbb{F}_q)$ MRMC

$\mathcal{C}_T := \{M_T : M \in \mathcal{C}\} \leq M_{m,n}(\mathbb{F}_q)$ MRMC.

- $\mathcal{C}$ MRD if and only if $\mathcal{C}_T$ MRD;
- $\mathcal{C}$ optimal anticode if and only if $\mathcal{C}_T$ optimal anticode.

MRD: an example

$\mathbb{F}_2 \subseteq \mathbb{F}_4 = \mathbb{F}_2(\alpha)$, $\alpha^2 = \alpha + 1$

$C = \langle(\alpha, 1)\rangle$

$d_{\min}(C) = 2$

MRD

$$1 = dim_{\mathbb{F}_4}(C) = n - d_{\min}(C) + 1$$

Not optimal vector anticode

$$1 = dim_{\mathbb{F}_4}(C) < 2 = \mathrm{maxr}(C)$$

Standard optimal anticodes

Standard optimal anticode

- $\mathcal{C} \leq M_{n,m}(\mathbb{F}_q)$, $n \leq m$, with the last $n - k$ rows equal to zero.
- $\mathcal{C} \leq M_{n,m}(\mathbb{F}_q)$, $n \geq m$, with the last $m - k$ columns equal to zero.

Standard optimal vector anticode

$k \in \{0, ..., m\}$, $C = \langle \bar{\mathbf{e}}_1, ..., \bar{\mathbf{e}}_k \rangle \leq (\mathbb{F}_{q^m})^n$.

Does the mapping preserve the properties?

$C \leq (\mathbb{F}_{q^m})^n$ VRMC, $d := d_{\min}(C)$, $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ be a basis of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$. $\Gamma(C) \leq M_{n,m}(\mathbb{F}_q)$ associated MRMC.

$n \leq m$

- C MRD $\Leftrightarrow$ $\Gamma(C)$ MRD;
- C optimal vector anticode $\Leftrightarrow$ $\Gamma(C)$ optimal anticode.

Does the mapping preserve the properties?

$C \leq (\mathbb{F}_{q^m})^n$ VRMC, $d := d_{\min}(C)$, $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ be a basis of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$. $\Gamma(C) \leq M_{n,m}(\mathbb{F}_q)$ associated MRMC.

$n > m$

- $C, \Gamma(C)$ MRD $\Leftrightarrow C = 0, (\mathbb{F}_{q^m})^n$;
- C optimal vector anticode and $\Gamma(C)$ optimal anticode $\Leftrightarrow$ $C = 0$.

de Seguins Pazzis

$m \neq n$
$M_{n,m}(\mathbb{F}_q)(V)$, for some $V \leq \mathbb{F}_q^{\min\{m,n\}}$.

$m = n$
$M_{n,m}(\mathbb{F}_q)(V), M_{n,m}(\mathbb{F}_q)(V)_T$, for some $V \leq \mathbb{F}_q^n$.

Classification of optimal vector anticodes

Ravagnani

Assume $n \leq m$

$C \leq (\mathbb{F}_{q^m})^n$ VRMC of dimension $k \leq m$

TFAE

- C optimal vector anticode;
- it has a bases of vectors with entries in $\mathbb{F}_q$;
- $C \sim \langle \overline{\mathbf{e}}_1, ..., \overline{\mathbf{e}}_k \rangle$.

Duality

MRMC

$\mathcal{C} \leq M_{n,m}(\mathbb{F}_q)$:

$$\mathcal{C}^{\perp} := \{M \in M_{n,m}(\mathbb{F}_q) : \mathrm{Tr}(MN_T) = 0, \ \forall N \in \mathcal{C}\}.$$

VRMC

$C \leq (\mathbb{F}_{q^m})^n$:

$$C^{\perp} := \{\overline{\mathbf{v}} \in \left(\mathbb{F}_{q^m}\right)^n : \overline{\mathbf{v}} \cdot \overline{\mathbf{u}} = 0, \ \forall \overline{\mathbf{u}} \in C\}.$$

Some properties of the dual

$$\dim_{\mathbb{F}_{q^m}}(C) + \dim_{\mathbb{F}_{q^m}}(C^\perp) = n \quad \forall C \leq (\mathbb{F}_{q^m})^n.$$

so...

$$\dim_{\mathbb{F}_q}(\Gamma(C)) + \dim_{\mathbb{F}_q}(\Gamma(C)^\perp) = mn.$$

Ravagnani

$$\dim_{\mathbb{F}_q}(\mathcal{C}) + \dim_{\mathbb{F}_q}(\mathcal{C}^\perp) = mn \quad \forall \mathcal{C} \leq M_{n,m}(\mathbb{F}_q).$$

Some properties of the dual

RAVAGNANI

$\mathcal{C}, \mathcal{D} \leq M_{n,m}(\mathbb{F}_q)$ MRMC

$(\mathcal{C}^\perp)^\perp = \mathcal{C}$

$(\mathcal{C} \cap \mathcal{D})^\perp = \mathcal{C}^\perp + \mathcal{D}^\perp$

$(\mathcal{C} + \mathcal{D})^\perp = \mathcal{C}^\perp \cap \mathcal{D}^\perp$

Some properties of the dual

$\mathcal{C} \leq M_{n,m}(\mathbb{F}_q)$ MRMC

Delsarte – Ravagnani

$\mathcal{C}$ MRD if and only if $\mathcal{C}^\perp$ is so.

Ravagnani

$\mathcal{C}$ optimal anticode if and only if $\mathcal{C}^\perp$ is so.

Ravagnani

If $\dim(\mathcal{C}) = k \max\{m, n\}$ we have an optimal anticode if and only if

$$\mathcal{C} + \mathcal{D} = M_{n,m}(\mathbb{F}_q)$$

for each $\mathcal{D} \leq M_{n,m}(\mathbb{F}_q)$ MRD such that $\dim(\mathcal{D}) = k + 1$.

SOME PROPERTIES OF THE DUAL

$C \leq (\mathbb{F}_{q^m})^n$ VRMC; Γ basis of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$. Is it true that

$$\Gamma(C)^\perp = \Gamma(C^\perp)??$$

Usually, the answer is negative, but it is still not time to surrender.

Don't give up! Orthogonal bases

$\mathbb{F}_q \subseteq \mathbb{F}_{q^m}$.

$$\Gamma = \{\gamma_1, ..., \gamma_k\}, \quad \Gamma' = \{\gamma'_1, ..., \gamma'_k\}$$

Γ, Γ' orthogonal, if and only if

$$\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\gamma_i \gamma'_j) = \delta_{i,j} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Given Γ there's a unique orthogonal basis Γ'

Orthogonal bases and duality: the final win

$C \leq (\mathbb{F}_{q^m})^n$ VRMC; Γ, Γ' orthogonal bases of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$. Then

$$\Gamma(C)^\perp = \Gamma'(C^\perp)$$

RAVAGNANI
$\mathcal{C} \leq M_{n,m}(\mathbb{F}_q)$ MRMC

$$d_{min}(\mathcal{C}^{\perp}) \leq \min\{m, n\} + 2 - d_{min}(\mathcal{C})$$

with equality if and only if $\mathcal{C}$ MRD.

Max rank and duality

Ravagnani

$\mathcal{C} \leq M_{n,m}(\mathbb{F}_q)$ MRMC

$$\mathrm{maxr}(\mathcal{C}) \geq \min\{m, n\} - \mathrm{maxr}(\mathcal{C}^{\perp})$$

with equality if and only if $\mathcal{C}$ optimal anticode.

(Distance + max rank) and duality

Ravagnani

$\mathcal{C} \leq M_{n,m}(\mathbb{F}_q)$ MRMC

$$d_{min}(\mathcal{C}) \leq \mathrm{maxr}(\mathcal{C}^\perp) + 1$$

with equality if and only if $\mathcal{C}$ is both MRD and optimal anticode (quite rare!).

Generalized weights

$C \leq (\mathbb{F}_{q^m})^n$ VRMC; let $n \leq m$.

v1 - Oggier - Sboui

$$w_i(C) = \min_D\{\max_v\{\dim(\mathrm{Supp}(\overline{\mathbf{v}})) : \overline{\mathbf{v}} \in D, \overline{\mathbf{v}} \neq \overline{\mathbf{0}}\} : D \leq C, \dim_{\mathbb{F}_{q^m}}(D) = i\}$$

for $i = 1, ..., \dim_{\mathbb{F}_{q^m}}(C)$.

Generalized weights

$C \leq (\mathbb{F}_{q^m})^n$ VRMC (let $n \leq m$).

$\forall D \leq (\mathbb{F}_{q^m})^n$, $D^* := D + \theta(D) + ... + \theta^{m-1}(D)$, where θ is the Frobenius endom.
D^* minimal $\mathbb{F}_{q^m}$-space containing D and fixed by Frobenius.

v2 - Ducoat

$$w_i(C) = \min_D\{\max_v\{\dim(\mathrm{Supp}(\overline{\mathbf{v}})) : \overline{\mathbf{v}} \in D^*, \overline{\mathbf{v}} \neq \overline{\mathbf{0}}\} : D \leq C, \dim_{\mathbb{F}_{q^m}}(D) = i\}$$

for $i = 1, ..., \dim_{\mathbb{F}_{q^m}}(C)$.

$C \leq (\mathbb{F}_{q^m})^n$ VRMC.

v3 - JURRIUS-PELLIKAAN

$$w_i(C) = \min\{\dim(\mathrm{Supp}(D)) : D \leq C, \dim_{\mathbb{F}_{q^m}}(D) = i\}$$

for $i = 1, ..., \dim_{\mathbb{F}_{q^m}}(C)$.

$C \leq (\mathbb{F}_{q^m})^n$ VRMC.

$\dim_{\mathbb{F}_{q^m}}(C) = k \leq m$

v4 - Ravagnani

$w_i(C) = \min\{\dim(A) : A \leq (\mathbb{F}_{q^m})^n, \text{ opt. vect. anticode} \dim_{\mathbb{F}_{q^m}}(C \cap A) \geq i\}$

for $i = 1, ..., \dim_{\mathbb{F}_{q^m}}(C)$.

$C \leq (\mathbb{F}_{q^m})^n$ VRMC.

v5 - RANDRIANARISOA

$$w_i(C) = \min\{\dim(A) : A \leq (\mathbb{F}_{q^m})^n,\ \text{Frobenius closed}\ \dim_{\mathbb{F}_{q^m}}(C \cap A) \geq i\}$$

for $i = 1, \ldots, \dim_{\mathbb{F}_{q^m}}(C)$.

Ravagnani

$C \leq M_{n,m}(\mathbb{F}_q)$

$$w_i(C) := \frac{1}{\max\{m,n\}} \min\{\dim(\mathcal{A}) : \mathcal{A} \leq M_{n,m}(\mathbb{F}_q) \text{ optimal anticode, } \dim(C \cap \mathcal{A}) \geq i\}$$

$i = 1, ..., \dim(C)$

GJLR

Invariant for equivalence.

Remember the classification of optimal anticodes:

$m \neq n$

$M_{n,m}(\mathbb{F}_q)(V)$, for some $V \leq \mathbb{F}_q^{\min\{m,n\}}$.

$n = n$

$M_{n,m}(\mathbb{F}_q)(V)$,$M_{n,m}(\mathbb{F}_q)(V)_T$, for some $V \leq \mathbb{F}_q^n$.

Moreover

$\dim(M_{n,m}(\mathbb{F}_q)(V)) = \max\{m, n\} \dim(V)$

This implies that for $i = 1, ..., \dim(\mathcal{C})$

$m \neq n$

$$w_i(\mathcal{C}) = \min\{\dim(V) : V \leq \mathbb{F}_q^{\min\{n,m\}} \dim(\mathcal{C}(V)) \geq i\}$$

$m = n$

$$w_i(\mathcal{C}) = \min\{\dim(V) : V \leq \mathbb{F}_q^n, \max\{\dim(\mathcal{C}(V)), \dim(\mathcal{C}_T(V))\} \geq i\}$$

Generalized weights: VRMC - MRMC

Ravagnani

Let $n \leq m$ and $C \leq (\mathbb{F}_{q^m})^n$ VRMC. We call, as usual, $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ a basis of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$.

$$w_i(C) = w_{mi-e}(\Gamma(C))$$

for $i = 1, ..., \dim_{\mathbb{F}_{q^m}}(C)$, $e = 0, ..., m-1$.

RELATIVE GENERALIZED WEIGHTS

KURIHARA - MATSUMOTO - UYEMATSU

$C \leq (\mathbb{F}_{q^m})^n$ VRMC.

$\forall D \leq C$, a proper subspace

$$w_i(C, D) = \min\{\dim(\mathrm{Supp}(V)) : V \leq (\mathbb{F}_{q^m})^n \text{ Frob. closed}, \dim_{\mathbb{F}_{q^m}}(C \cap V) - \dim_{\mathbb{F}_{q^m}}(D \cap V) \geq i\}$$

for $i = 1, ..., \dim_{\mathbb{F}_{q^m}}(C) - \dim_{\mathbb{F}_{q^m}}(D)$.

Relative generalized weights

Martínez-Peñas

$\mathcal{C} \leq M_{n,m}(\mathbb{F}_q)$ MRMC.

$\forall \mathcal{D} \leq \mathcal{C}$, a proper subspace.
$M_{n,m}(\mathbb{F}_q)(V)^{colsp} := \{M \in M_{n,m}(\mathbb{F}_q) : colsp(M) \leq V\}$

$$w_i(\mathcal{C}, \mathcal{D}) = \min\{ \dim(\mathcal{C} \cap M_{n,m}(\mathbb{F}_q)(V)^{colsp}) - \dim(\mathcal{D} \cap M_{n,m}(\mathbb{F}_q)(V)^{colsp}) \geq 0, V \leq \mathbb{F}_q^n\}$$

for $i = 1, ..., \dim(\mathcal{C}) - \dim(\mathcal{D})$.

Thank you for your attention!